Math3310 Tutorial 1

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1 Some classes of first order ODE

1.1 Linear

A linear first-order ODE is of the form

$$y'(t) = g(t)y(t) + h(t), \quad y(t_0) = y_0$$

Rewrite the problem in the form:

$$y'(t) - g(t)y(t) = h(t)$$

The strategy is to find an integrating factor $u(t)$ and multiply both sides by it.

$$uy' - ugy = uh \tag{1}$$

Why did we do this? Because of the product rule. Let $z(t) = u(t)y(t)$ then $z' = uy' + u'y$. So if we can find a u such that $u' = -ug$, then (1) can be simplified to $z' = uh$, which can be further solved by direct integration. In all,

$$\begin{aligned}
u(t) &= \exp(-\int_{s_0}^{t} g(s)ds) \\
z(t) &= \int_{s_0}^{t} u(s)h(s)ds + C \\
y(t) &= \left[\int_{s_0}^{t} u(s)h(s)dt\right] /u(t) + C/u(t)
\end{aligned} \tag{2}$$

How to choose s_0, and is it important how to choose s_0? I will discuss this in the tutorial.

Here are some exercises.

1. $y' = 2ty + t, \quad y(0) = 1$
2. $(1 + t^2)y' = 2ty + 1 + t^2 \quad y(1) = 2$
3. $y' = 1 + x + y + xy, \quad y(0) = 0$

1.2 Separable

A separable first-order ode has the form

$$y'(t) = g(t)h(y), \quad y(t_0) = y_0$$

or another not rigorous form is

$$H(y)dy = g(t)dt, \quad y(t_0) = y_0$$

which can be easily solved by direct integration and then adding a constant whose value is determined by the initial condition.

$$\int H(y)dy = \int g(t)dt + C$$

1.2.1 One homogeneous variant

An interesting variant is the homogeneous first-order ode in the following form.

$$y'(t) = g(\frac{y}{t}), \quad y(t_0) = y_0 \text{ and } t > 0$$

To solve it, we introduce a variable $v(t) = \frac{y}{t}$ and have $y'(t) = tv'(t) + v(t) = g(v)$.
Cleaning this up, we obtain a separable ODE.

$$v'(t) = \frac{g(v) - v}{t}$$

Here are some exercises.

1. solve $y' = 1 + \frac{y}{t} + (\frac{y}{t})^2$ for $t > 0$ and y(1) = 1
2. solve $y' = \frac{y^2 - ty + t^2}{t^2}$, $y(1) = 2$

1.3 Extra Topics

1.3.1 Exact Equations

an exact equation has the form:

$$f(t,y)dt + g(t,y)dy = 0$$

where f and g satisfy $f_y = g_t$. Recall that for a smooth function $h(t,y)$, we always have $h_{ty} = h_{yt}$, so we may guess that there exists a function $h(t,y)$ such that $dh = h_t dt + h_y dy$ and $h_t(t,y) = f(t,y)$ and $h_y(t,y) = g(t,y)$.

How to find such h? First, integrating $f(t,y)$ w.r.t t, to "eliminates" the effect of t. We have

$$h(t,y) = \int f(t,y)dt + C(y) = F(t,y) + C(y)$$

Doing the same thing to y and $g(t,y)$, we have

$$h(t,y) = \int g(t,y)dy + D(t) = G(t,y) + D(t)$$

Last find $C(y)$ and $D(t)$ by requiring

$$F(t,y) + C(y) = G(t,y) + D(t)$$

Substituting these into the equations, we have

$$dh = f(t,y)dt + g(t,y)dy = 0$$

so in this ode, $y(t)$ is determined implicitly by t via the equation

$$F(t,y) + C(y) = E$$

where E is a constant determined by the initial condition. Here are some exercises.

1. solve $(y\cos(t) + 2te^y)dt + (\sin(t) + t^2 e^y + 2)dy = 0$, $y(0) = 1$
2. solve $2tydt + (2y + t^2)dy = 0$, $y(1) = 2$

1.3.2 Bernoulli

The last type I would like to mention is the Bernoulli first-order ode.

$$y'(t) = g(t)y(t) + h(t)\left[y(t)\right]^n$$

Here n does not equal 1. Otherwise, it is the aforementioned separable ODE.

The key idea is to make a suitable substitution. Remind that $(y^k)' = ky^{k-1}y'$. So multiply both sides by y^{-n} and let $v = y^{1-n}$, we have

$$\frac{1}{1-n}v' = g(t)v(t) + h(t)$$

which is a simple linear ODE.

Here are some exercises.

1. solve $y' = 5y - 5ty^3$, $y(0) = \frac{1}{10}$
2. solve $y' + ty = ty^4$, $y(0) = 2$

2 linear second order ODE

2.1 constant coefficients

First, we discuss the easiest case.

$$ay'' + by' + cy = 0$$

where a, b, c are constant, $a \neq 0$ and the equation is homogeneous.

Let $y(x) = e^{rx}$ and the question would be simplified to a quadratic equation in one Unknown $ar^2 + br + c = 0$. There are three possible cases.

1. $b^2 - 4ac > 0$. There are two distinct real roots r_1 and r_2. The general solution is $y(x) = \alpha_1 e^{r_1 x} + \alpha_2 e^{r_2 x}$ and α_1 and α_2 are determined by two conditions of y and its derivatives.

2. $b^2 - 4ac > 0$. There are two distinct complex roots $\alpha \pm i\beta$. The general solution is $y(x) = \alpha_1 e^{\alpha + i\beta x} + \alpha_2 e^{\alpha - i\beta x}$ which can be written as linear combination of $e^{\alpha x}\sin(\beta x)$ and $e^{\alpha x}\cos(\beta x)$

3. $b^2 - 4ac = 0$. There is only one repeated root r. So one of its solutions is $y_1 = e^{rx}$. What about another one y_2? Since y_2 must not be constant multiple of y_1, we may assume $y_2 = v(x) \cdot e^{rx}$ and substitute it into the ode, we would obtain $v'' = 0$, so a choice of v is $v(x) = x$ and $y_2 = xe^{rx}$.

What if the equation is not homogeneous?

$$ay'' + by' + cy = f(x)$$

If we find a particular solution $L(x)$ to this ode, then the general solution is $y(x) = L(x) + \alpha_1 y_1(x) + \alpha_2 y_2 x$. So how to find the $L(x)$?

1. Guess.

2. Variation of parameter. Detail would be discussed in the class. Here we give out the conclusion.

   Theorem 1 *Consider the ODE*

   $$ay'' + by' + cy = f(x)$$

   Let y_1 and y_2 be a fundamental set of solutions of its homogeneous type, then a particular solution is given by

   $$L(x) = -y_1(x)\int \frac{y_2(x)f(x)}{a(x)\cdot W(y_1, y_2)}dx + y_2(x)\int \frac{y_1(x)f(x)}{a(x)\cdot W(y_1, y_2)}dx$$

   where $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ is called Wronskian.

Here are some exercises.

1. Find a general solution to $y'' + 4y = sin(x)$

2. Find a general solution to $y'' + 2y' + 4y = e^x$

3. Find a general solution to $y'' - 2y' + y = \frac{e^x}{x^2+1}$

3 Fourier series and PDE

3.1 Eigenvalue problems and motivation

Following three ODEs are called *eigenproblems.*

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0, \tag{3}$$
$$x'' + \lambda x = 0, \quad x'(a) = 0, \quad x'(b) = 0, \tag{4}$$
$$x'' + \lambda x = 0, \quad x(a) = x(b), \quad x'(a) = x'(b). \tag{5}$$

A number λ is called an *eigenvalue* of (3) (resp. (4) or (5)) if and only if there exists a nonzero (not identically zero) solution to (3) (resp. (4) or (5)) given that specific λ. A nonzero solution is called a corresponding *eigenfunction.* Eigenvectors for two distinct eigenvalues of a symmetric matrix are orthogonal. There is an analogue of this property about eigenfunctions. Before introducing it, we should introduce the concept of inner product in the vector space of functions, which is defined as

$$\langle f, g\rangle_1 \triangleq \int_a^b f(t)g(t)dt$$

A useful observation is as follows:

Theorem 2 *Suppose that $x_1(t)$ and $x_2(t)$ are two eigenfunctions of the problem* (3), (4) ***or*** (5) *for two different eigenvalues λ_1 and λ_2. Then they are orthogonal w.r.t the inner prodect $\langle , \rangle_1$, namely*

$$\int_a^b x_1(t)x_2(t)dt = 0$$

Proof would be given in the tutorial.

This property is called *orthogonality of eigenfunctions.* An important example is given in the class and $a = -\pi, b = \pi$ in this case.

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \delta_m(n)$$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = \delta_m(n)$$

$$\int_{-\pi}^{\pi} \cos(mt)\sin(nt)dt = \delta_m(n)$$

3.2 2L-periodic functions

In the class, all content about Fourier series is related to a set of eigenfunctions $\{\cos(nx), \sin(nx)\}$. But when the period of function switches from 2π to a more general case $2L$, all eigenfunctions $f_n(x)$ would change and can be easily found by solving equations (3) (resp. (4) or (5)). A shortcut is to directly compute λ such that $\cos(\lambda x) = \cos(\lambda(x + 2L))$ so the new set of eigenfunctions is

$$\left\{\cos(\frac{\pi}{L}nx), \sin(\frac{\pi}{L}nx)\right\}$$

Next is to compute the fourier series of a $2L$-periodic functions. Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{L}nx) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi}{L}nx)$$

then

$$\begin{aligned}
\int_{-L}^{L} f(x)1dx &= \int_{-L}^{L} a_0 dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos(\frac{\pi}{L}nx)dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin(\frac{\pi}{L}nx)dx \\
&= 2L \cdot a_0 \\
\int_{-L}^{L} f(x)\cos(\frac{\pi}{L}kx)dx &= \int_{-L}^{L} a_0 \cos(\frac{\pi}{L}kx)dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos(\frac{\pi}{L}nx)\cos(\frac{\pi}{L}kx)dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin(\frac{\pi}{L}nx)\cos(\frac{\pi}{L}kx)dx \\
&= a_k \int_{-L}^{L} \cos(\frac{\pi}{L}kx)\cos(\frac{\pi}{L}kx)dx \\
&= L \cdot a_k \\
\int_{-L}^{L} f(x)\sin(\frac{\pi}{L}kx)dx &= \int_{-L}^{L} a_0 \sin(\frac{\pi}{L}kx)dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos(\frac{\pi}{L}nx)\sin(\frac{\pi}{L}kx)dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin(\frac{\pi}{L}nx)\sin(\frac{\pi}{L}kx)dx \\
&= b_k \int_{-L}^{L} \sin(\frac{\pi}{L}kx)\sin(\frac{\pi}{L}kx)dx \\
&= L \cdot b_k
\end{aligned}$$